Episode 7

Energy Relations for Conservative Systems of Particles

ENGN0040: Dynamics and Vibrations
Allan Bower, Yue Qi

School of Engineering
Brown University

Topics for todays class

Energy relations for conservative systems of particles

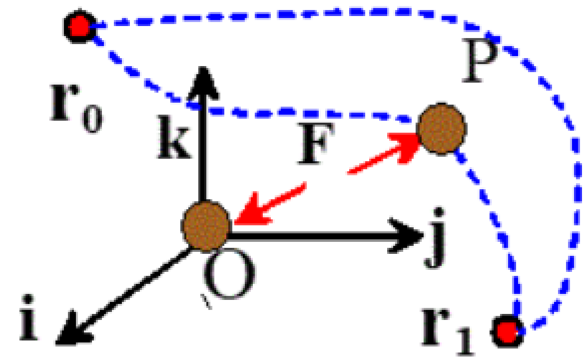
1. Conservative, non-conservative and workless forces
2. Potential energy of a conservative force
3. Work-Power-PE-KE relation for a conservative system of particles
4. Applications

4.2 Energy relations for conservative systems of particles

4.2.1 Conservative, non-conservative and workless forces

Recall: Work done by a force

$$W = \int_{\underline{r}_0}^{\underline{r}_1} \underline{F} \cdot d\underline{r}$$



Definition For a "conservative" force W is equal for all paths from $\underline{r}_0 \to \underline{r}_1$

Conservative
Gravity
Electrostatic forces
Inter-molecular forces
Forces exerted by springs

Non-Conservative
Friction
Air resistance

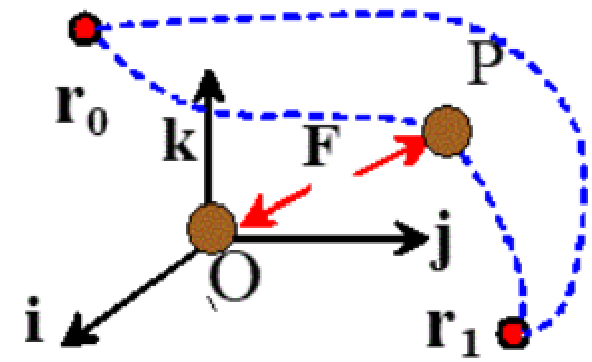
Workless
Lift force
Reaction forces
Lorenz force

$W = 0$ for a "workless" force

4·2·2 Potential energy of a conservative force

Definition

$$\tilde{U}(\underline{r}) = -\int_{\underline{r}_0}^{\underline{r}} \underline{F} \cdot d\underline{r} + C$$



$\underline{r}_0$ & C are arbitrary : choose them to make $\tilde{U}$ simple

Inverse relation

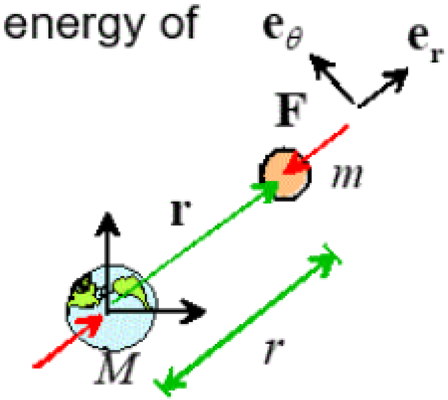
$$\underline{F} = -\nabla\tilde{U} = -\left\{ \frac{\partial \tilde{U}}{\partial x}\underline{i} + \frac{\partial \tilde{U}}{\partial y}\underline{j} + \frac{\partial \tilde{U}}{\partial z}\underline{k} \right\}$$

Note − signs !

4.2.3: Example: Calculating the potential energy of gravity (between planets)

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r$$

$$G = 6.674\times10^{-11}\ m^3 kg^{-1} s^{-2}$$



Formula

$$U = -\int_{\underline{r}_0}^{\underline{r}} \underline{F}\cdot d\underline{r} + C$$

Do integral in $\{\underline{e}_r, \underline{e}_\theta\}$ basis

$$U(\underline{r}) = -\int_{r_0\underline{e}_r}^{r\underline{e}_r} -\frac{GMm}{r^2}\underline{e}_r \cdot dr\,\underline{e}_r + C$$

$$= \int_{r_0}^{r} \frac{GMm}{r^2}dr + C = \left[-\frac{GMm}{r}\right]_{r_0}^{r} + C$$

$$U = -\frac{GMm}{r} + \boxed{\frac{GMm}{r_0} + C}$$

Choose $C = -GMm/r_0$

$$\boxed{U = -\frac{GMm}{r}}$$

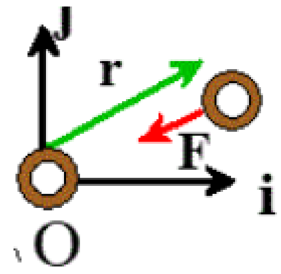
$$r = \sqrt{x^2+y^2+z^2}$$

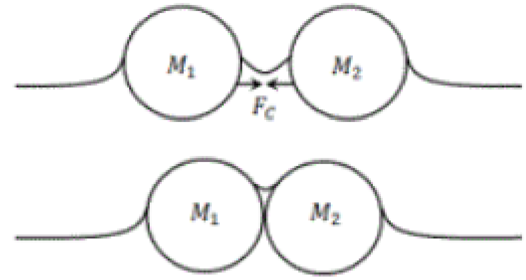
4.2.4: Example: The potential energy of two neighboring Cheerios floating in milk is

$$U \approx E_0 \log\left(\frac{r}{L_0}\right) \qquad r = \sqrt{x^2 + y^2}$$

where E_0, L_0 are constants

Find a formula for the force acting between them





Formula $\underline{F} = -\nabla U$

Use chain rule

$$\underline{F} = -\left\{ \frac{\partial U}{\partial r}\frac{\partial r}{\partial x}\underline{i} + \frac{\partial U}{\partial r}\frac{\partial r}{\partial y}\underline{j} \right\}$$

Note $\displaystyle \frac{\partial r}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}\,2x = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$

$$\underline{F} = \underbrace{\frac{\partial U}{\partial r}}_{\text{magnitude}} \underbrace{\left\{ -\frac{(x\underline{i} + y\underline{j})}{r} \right\}}_{\text{Unit vector - direction}} = \frac{\partial U}{\partial r}\left(-\frac{\underline{r}}{|\underline{r}|} \right)$$

For cheerios $\frac{\partial U}{\partial r} = \frac{E_0}{r} \Rightarrow \boxed{\underline{F} = \frac{E_0}{|\underline{r}|}\left(\frac{-\underline{r}}{|\underline{r}|}\right)}$

In general for a PE of form $U(r)$

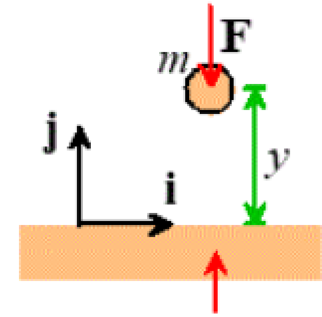
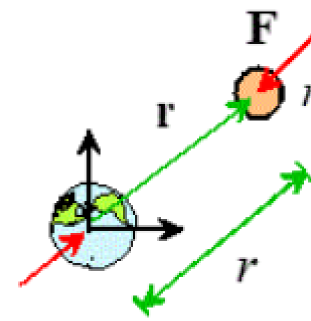
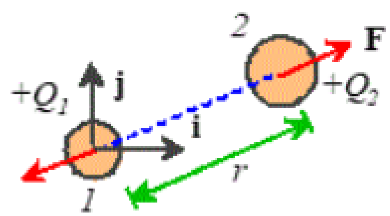
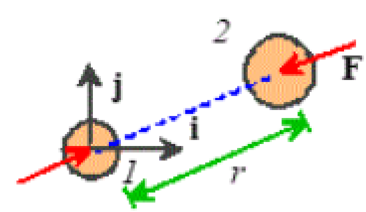
(1) Magnitude is $\frac{\partial U}{\partial r}$

(2) Direction is towards or away from origin

$\frac{\partial U}{\partial r} > 0 \Rightarrow$ force acts towards origin

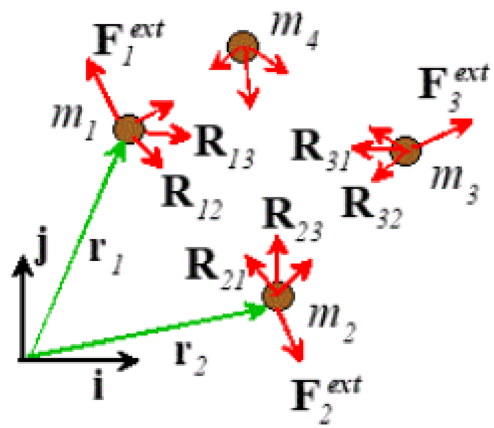
$\frac{\partial U}{\partial r} < 0 \Rightarrow$ force acts away from origin

4.2.5: Table of potential energies for common forces

Type of force	Force vector	Potential energy	
Gravity acting on a particle near earths surface	$\mathbf{F} = -mg\mathbf{j}$	$U = mgy$	
Gravitational force exerted on mass m by mass M at the origin	$\mathbf{F} = -\dfrac{GMm}{r^3}\mathbf{r}$	$U = -\dfrac{GMm}{r}$	
Force exerted by a spring with stiffness k and unstretched length L_0	$\mathbf{F} = -k(r - L_0)\dfrac{\mathbf{r}}{r}$	$U = \dfrac{1}{2}k(r - L_0)^2$	
Force acting between two charged particles	$\mathbf{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon r^3}\mathbf{r}$	$U = \dfrac{Q_1 Q_2}{4\pi\varepsilon r}$	
Force exerted by one molecule of a noble gas (e.g. He, Ar, etc) on another (Lennard Jones potential). a is the equilibrium spacing between molecules, and E is the energy of the bond.	$\mathbf{F} = 12\dfrac{E}{a}\left[\left(\dfrac{a}{r}\right)^{13} - \left(\dfrac{a}{r}\right)^{7}\right]\dfrac{\mathbf{r}}{r}$	$U = E\left[\left(\dfrac{a}{r}\right)^{12} - 2\left(\dfrac{a}{r}\right)^{6}\right]$	

4.2.6 Energy equation for a conservative system

$\mathbf{R}_{ij}$	Force exerted on particle i by particle j
$\mathbf{F}_i^{ext}$	External force on particle i
$\mathbf{r}_i$	Position of particle i
$\mathbf{v}_i$	Velocity of particle i



$\underline{R}_{ij}$: "Internal" forces $\quad$ $\underline{F}_i^{ext}$: "external" forces

Definition $\quad$ system is conservative $\Leftrightarrow$ all $\underline{R}_{ij}$ are conservative

Let $$\boxed{\underline{R}_{ij} = -\frac{\partial U_{ij}}{\partial \underline{r}_i}}$$

Definitions

① Total PE of system $U^{TOT} = \sum_{\substack{\text{internal} \\ \text{forces}}} U_{ij}$

② Total KE of system $T^{TOT} = \sum_{\text{particles}} \frac{1}{2} m_i \, |\underline{v}_i|^2$

③ Power of external forces $\mathcal{P}^{ext} = \sum_{\substack{\text{ext} \\ \text{forces}}} \underline{F}_i^{ext} \cdot \underline{v}_i$

④ Total external work done on system

$$\Delta W^{ext} = \int_{t_0}^{t_1} \mathcal{P}^{ext} \, dt$$

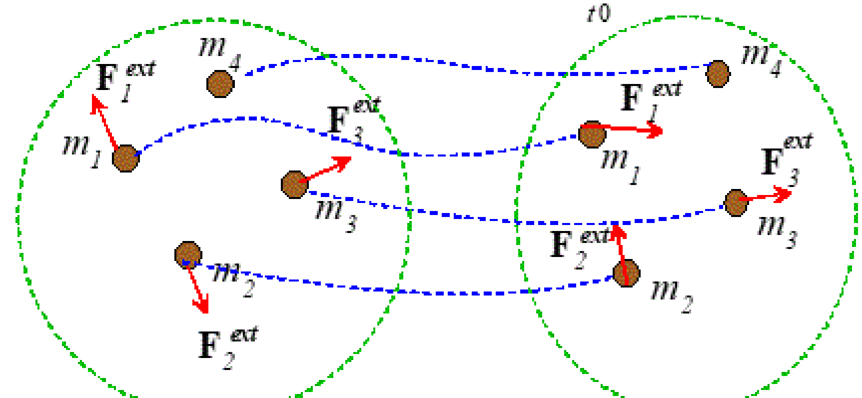
Energy Equations

External Power $P^{ext}(t)$

External work $\Delta W^{ext} = \int_{t0}^{t1} P^{ext}(t)dt$

Power-Energy relation

$$\boxed{P^{ext} = \frac{d}{dt}\left\{ U^{TOT} + T^{TOT} \right\}}$$



$t = t_0$ | $t = t_1$

Total KE T_0^{TOT} | Total KE T_1^{TOT}

Total PE U_0^{TOT} | Total PE U_1^{TOT}

Work-energy relation

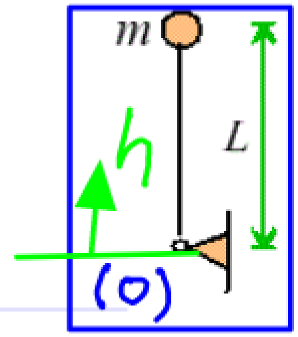
$$\boxed{\Delta W^{ext} = U_1^{TOT} + T_1^{TOT} - (U_0^{TOT} + T_0^{TOT})}$$

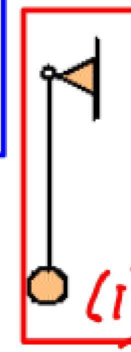
Special Case $\Delta W^{ext} = 0$

$$\boxed{U_1^{TOT} + T_1^{TOT} = U_0^{TOT} + T_0^{TOT}}$$

"Total energy is conserved"

4.2.7: Example: The pendulum is stationary in its upright configuration. Following a small disturbance it falls over. Calculate the magnitudes of the velocity and acceleration of the mass when it reaches its lowest point.





Approach:

(1) System = earth + pendulum

(2) Conservative, no ext forces $\Rightarrow \Delta W^{ext} = 0$

Hence $T_1 + U_1 = T_0 + U_0$

State (0) $T_0 = 0 \quad U_0 = mgL$

State (1) $T_1 = \frac{1}{2} m V^2 \quad U_1 = -mgL$

$$\Rightarrow \frac{1}{2} m V^2 - mgL = mgL \Rightarrow \boxed{V = 2\sqrt{gL}}$$

Acceleration Circular motion $\Rightarrow \underline{a} = \underbrace{\frac{dV}{dt}}_{=0} \underline{t} + \frac{V^2}{R} \underline{n}$ $R = L$

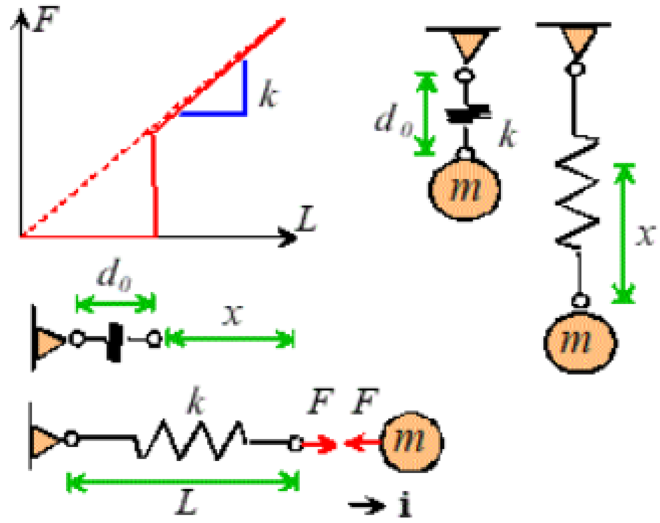
$$\Rightarrow |\underline{a}| = \frac{V^2}{L} = \frac{4gL}{L} \Rightarrow \boxed{|\underline{a}| = 4g}$$

4.2.8: Example: A spring has the force-length relation shown.

(a) Find a formula for its potential energy

(b) A mass $m > kd_0/g$ is suspended from the spring. Find the value of extension x in the static equilibrium configuration

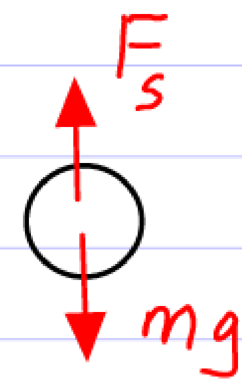
(c) The mass released from rest with $x=0$. Find the maximum value of x and the acceleration of the mass at this instant.



(a) $\bar{U} = -\int_{\underline{r}_0}^{\underline{r}_1} \underline{F} \cdot d\underline{r} \qquad \underline{F} = -k(x+d_0)\,\underline{i}$

$$\Rightarrow \bar{U} = \int_0^x k(x+d_0)\,dx \Rightarrow \boxed{\bar{U} = \frac{1}{2}kx^2 + kd_0 x}$$

(b) Statics



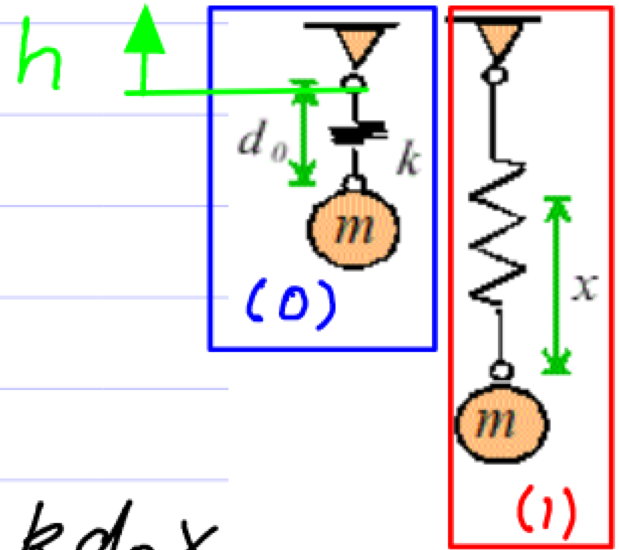
$$F_s = mg \Rightarrow k(x+d_0) = mg$$

$$\Rightarrow \boxed{x = \frac{mg}{k} - d_0}$$

(c) System = earth + spring + mass

Conservative, no ext forces

$$T_1 + U_1 = T_0 + U_0$$



(0) $T_0 = 0$ $\quad U_0 = -mg\,d_0$

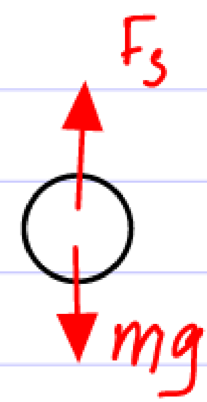
(1) $T_1 = 0$ $\quad U_1 = -mg\,(d_0 + x) + \frac{1}{2}kx^2 + k d_0 x$

$$\Rightarrow \quad -mg\,(\cancel{d_0} + \underset{1}{\cancel{x}}) + k\cancel{x}\left(\frac{x}{2} + d_0\right) = \cancel{-mg\,d_0}$$

$$\Rightarrow \quad \boxed{x = 2\left(\frac{mg}{k} - d_0\right)}$$

Twice static deflection

<u>Acceleration</u> Use $\underline{F} = m\underline{a}$



$$ma = F_s - mg = k(x + d_0) - mg$$

$$\Rightarrow \quad \boxed{a = g - \frac{k d_0}{m}}$$

Spring Drop Experiment

Coiled spring
h = 22.5cm

33.5cm

Static equilibrium
h = 56cm

33cm

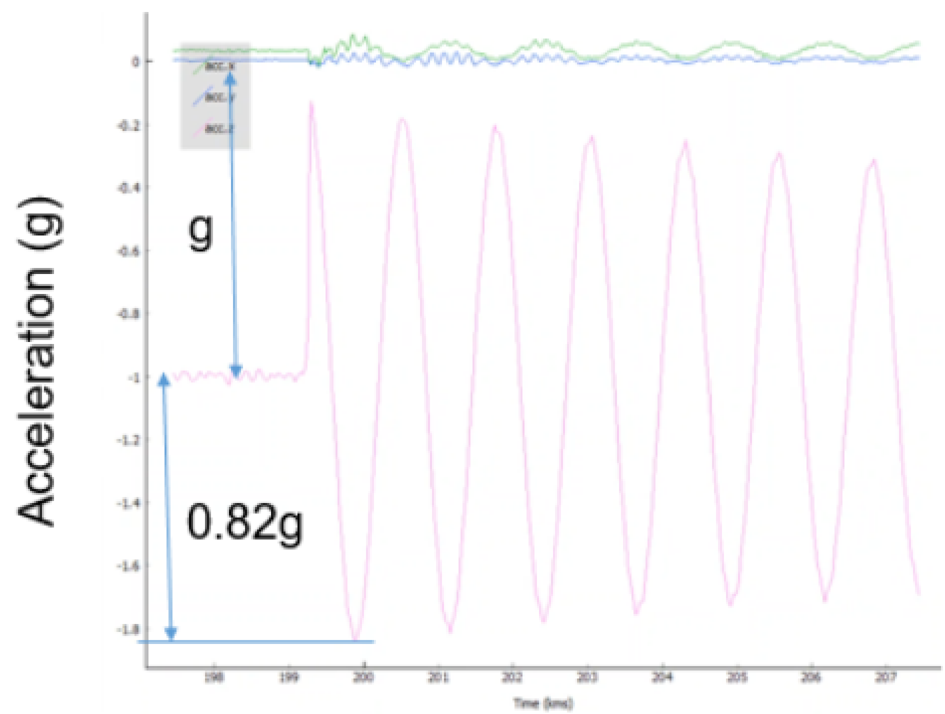
Max deflection
h = 89cm

Predicted
to be equal

Prediction

Spring stiffness: 3.5 N/m
Spring length at zero force 6.5cm
Total mass 142.8 grams

$$a = g - kd_0 / m = 0.84g$$



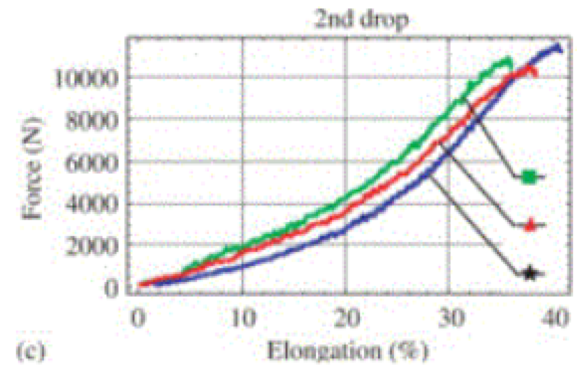
4.2.8: Example: A dynamic climbing rope has a force-elongation relation that can be approximated by $F = F_0 \varepsilon^{3/2}$

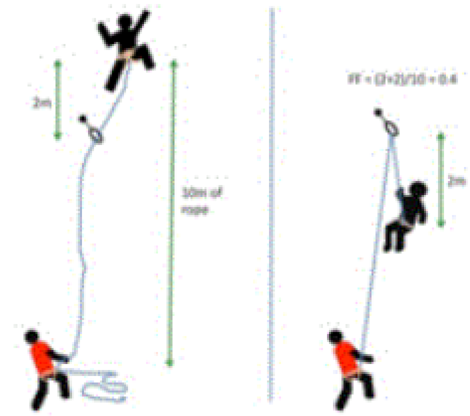
(a) Find a formula for the potential energy of a rope with length *L*

(b) A climber with mass *m* is tethered by a rope with length *L*. The climber falls a distance *h* before the rope begins to stretch, and is brought to rest as the rope stretches. Show that the fractional change in length of the rope ε is related to the fall factor *f=h/L* by

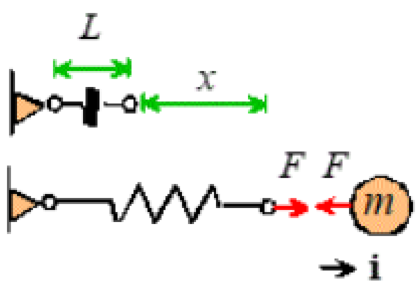
$$f = \frac{2F_0}{5mg}\varepsilon^{5/2} - \varepsilon$$

(c) Find the value of F_0 necessary to pass the UIAA standard





(a) Think of rope as nonlinear spring



$$\underline{F} = -F_0\, \varepsilon^{3/2} \qquad \varepsilon = x/L$$

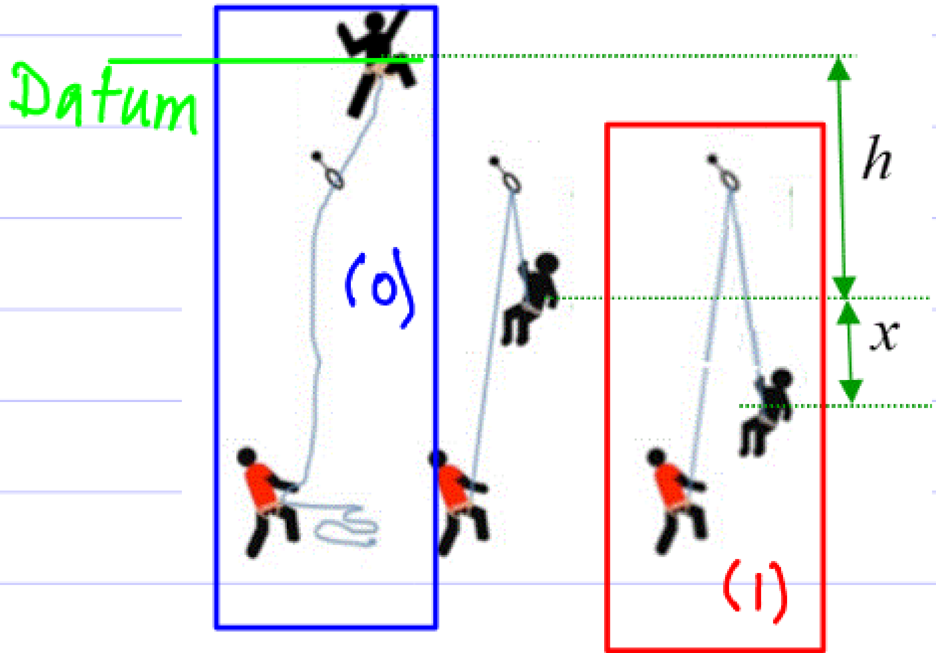
$$U = -\int_{\underline{r}_0}^{\underline{r}} \underline{F} \cdot d\underline{r} = \int_0^x F_0 \left(\frac{x}{L}\right)^{3/2} dx$$

$$\Rightarrow \boxed{U = \frac{2}{5} F_0 L \left(\frac{x}{L}\right)^{5/2} = \frac{2}{5} F_0 L \varepsilon^{5/2}}$$

(b) System = earth + rope + climber

Conservative, no ext force

$$T_1 + U_1 = T_0 + U_0$$



(0) $T_0 = 0 \qquad U_0 = 0$

(1) $T_1 = 0$

$$U_1 = -mg(h+x) + \frac{2}{5} F_0 L \left(\frac{x}{L}\right)^{5/2}$$

$$\Rightarrow \quad mgh = \frac{2}{5} F_0 L \left(\frac{x}{L}\right)^{5/2} - mgx$$

$$\Rightarrow \quad \frac{h}{L} = \frac{2}{5} \frac{F_0}{mg} \left(\frac{x}{L}\right)^{5/2} - \frac{x}{L}$$

$$\Rightarrow \quad \boxed{f = \frac{2}{5} \frac{F_0}{mg} \varepsilon^{5/2} - \varepsilon}$$

(c) UIAA standard

for $m = 80$ kg, $f = 1.75$: (a) $\varepsilon \le 0.4$

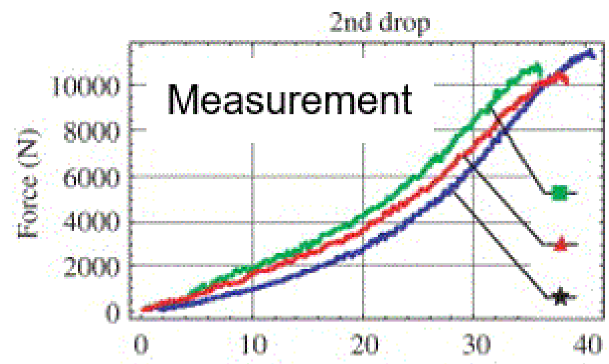
(b) Force in rope < 12 kN

We have $f = \frac{2}{5} \frac{F_0}{mg} \varepsilon^{5/2} - \varepsilon$

Use (a) $\Rightarrow$ $\boxed{F_0 = \frac{5}{2}(f+\varepsilon) \frac{mg}{\varepsilon^{5/2}} = 42 \text{ kN}}$

Check (b) $F = F_0 \varepsilon^{3/2} \simeq 11$ kN ✓

Postscript: ropes in example just pass standard



(c)